

LOSS OF CONTINUITY OF THE SOLUTION MAP FOR THE EULER EQUATIONS IN α -MODULATION AND HÖLDER SPACES

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ABSTRACT. We study the incompressible Euler equations in the α -modulation $M_{p,q}^{s,\alpha}$ and Hölder $C^{1+\sigma}$ spaces on the plane. We show that for these spaces the associated data-to-solution map is not continuous on bounded sets.

1. INTRODUCTION

In this paper we study the Cauchy problem for the non-periodic Euler equations of incompressible hydrodynamics

$$(E) \quad \begin{aligned} u_t + u \cdot \nabla u + \nabla p &= 0, & t \geq 0, x \in \mathbb{R}^n \\ \operatorname{div} u &= 0 \\ u(0) &= u_0 \end{aligned}$$

with initial data in the α -modulation spaces. In particular, our results apply to the Besov spaces including the classical Hölder-Zygmund spaces. According to the standard notion of well-posedness due to Hadamard a Cauchy problem is said to be locally (in time) well-posed in a Banach space X if given any initial data u_0 in X there is a time $T > 0$ and a unique solution u in a Banach space $Y \subset C([0, T], X)$ which depends continuously on the initial data. Otherwise the Cauchy problem is said to be locally ill-posed.

Ill-posedness results establishing loss of continuity of the solution map $u_0 \rightarrow u$ for the Euler equations in the C^1 space and the borderline Besov space $B_{\infty,1}^1$ have been proved recently in [18]. Here we refine the techniques of that paper to obtain ill-posedness results of this type in α -modulation spaces $M_{p,q}^{s,\alpha}$ for $1 < s < 2$ and in $C^{1+\sigma}$ for $0 < \sigma < 1$. More precisely, following the approach of Bourgain and Li [3] we construct a Lagrangian flow with a large gradient and then choose a suitable high-frequency perturbation of the initial vorticity to show that the assumption of continuity of the solution map $u_0 \rightarrow u$ in the above spaces necessarily leads to a contradiction with the results of Kato and Ponce [13, 14]. We will work with the vorticity equations which in two dimensions assume the form

$$(1.1) \quad \begin{aligned} \omega_t + u \cdot \nabla \omega &= 0, & t \geq 0, x \in \mathbb{R}^2 \\ u &= K * \omega = \nabla^\perp \Delta^{-1} \omega \\ \omega(0) &= \omega_0 \end{aligned}$$

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where

$$K(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2} \quad \text{and} \quad \nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right)$$

denote the Biot-Savart kernel and the symplectic gradient, respectively.

The first rigorous results on the Cauchy problem for the incompressible Euler equations go back to Gyunter [10], Lichtenstein [15] and Wolibner [23]. A survey of those and numerous further results can be found for example in Majda and Bertozzi [16], Constantin [5] or Bahouri, Chemin and Danchin [1]. For the most recent progress on local ill-posedness in borderline spaces such as C^1 , $W^{n/p+1,p}$, $B_{p,q}^{n/p+1}$ as well as C^k , $C^{k-1,1}$ for integer $k \geq 1$ we refer to the papers of Bourgain and Li [3, 4], Elgindi and Masmoudi [8] and the authors [18]. For earlier results in C^σ with $0 < \sigma < 1$, $B_{p,\infty}^s$ for $s > 0$, $p > 2$ and $s > n(2/p - 1)$, $1 \leq p \leq 2$ or the logarithmic Lipschitz spaces $\log \text{Lip}^\alpha$ for $0 < \alpha < 1$ we refer to Bardos and Titi [2], Cheskidov and Shvydkoy [6] and the authors [17].

Our main goal is to prove the following result.

Main Theorem. *Let $0 < \sigma < 1$, $0 < \alpha \leq 1$, $2 \leq p \leq \infty$, $1 \leq q \leq \infty$ and suppose that $M_{p,q}^{1+\sigma,\alpha}$ is continuously embedded in C^1 . Then the solution map of the incompressible Euler equations (E) is not continuous on bounded subsets of $M_{p,q}^{1+\sigma,\alpha}$.*

Thus, the Euler equations are in general locally ill-posed in α -modulation spaces in the sense of Hadamard given above.¹ For the definition of the α -modulation spaces see Section 2 below.

Remark 1. Observe that $M_{p,q}^{s,1}$ coincides with the usual Besov space $B_{p,q}^s$. Therefore, somewhat surprisingly, Theorem 1 also yields ill-posedness (in the sense that the data-to-solution map loses its continuity properties) even in the classical Hölder spaces $B_{\infty,\infty}^{1+\sigma} = C^{1+\sigma}$ for $0 < \sigma < 1$.

Continuous dependence results for the Euler equations (in the strong topology) have been obtained for initial data in Sobolev spaces H^s and more generally $W^{s,p}$ with $s > n/p + 1$ for example in Ebin and Marsden [8], Kato and Lai [12] and Kato and Ponce [14]. However, this is a rather difficult part of the local well-posedness theory which has not yet been satisfactorily resolved.

Remark 2. A different mechanism involving a gradual loss of regularity of the solution map is described by Morgulis, Shnirelman and Yudovich [19].

Remark 3. In this context it is also worth pointing out that neither for the critical Besov space $B_{\infty,1}^1(\mathbb{R}^n)$ nor for the space $B_{p,1}^{1+p/n}(\mathbb{R}^n)$ are the Euler equations strongly ill-posed in the sense of Bourgain and Li [3]. This can be seen by examining the arguments given in Pak and Park [20] and Vishik [22].

Our general strategy will be similar to that employed in [18] which we will use as the main reference. The remainder of the paper is organized as follows. In Section 2 we describe the general set up and prove several technical lemmas. The whole of Section 3 is then devoted to the proof of Theorem 1. Although the constructions in Sections 2 and 3 will be carried out in 2D they can be readily adapted to the 3D case. Rather than doing that in Section 4 we give a direct proof of ill-posedness in C^{1+s} by using a 3D shear flow argument.

¹For example, we have $M_{\infty,q}^{1+\sigma,\alpha} \subset C^1$ whenever $\sigma > 2(1 - \alpha)(1 - 1/q)$.

2. BASIC SETUP: VORTICITY AND LAGRANGIAN FLOW

We first recall the definition of α -modulation spaces. For a more detailed account the reader is referred for example to [9]. A countable set \mathcal{Q} of subsets $Q \subset \mathbb{R}^n$ is called an admissible covering if $\mathbb{R}^n = \bigcup_{Q \in \mathcal{Q}} Q$ and if there is $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq n_0$ for all $Q \in \mathcal{Q}$. Let

$$\begin{aligned} r_Q &= \sup\{r \in \mathbb{R} : B(c_r, r) \subset Q, c_r \in \mathbb{R}^n\} \\ R_Q &= \inf\{R \in \mathbb{R} : Q \subset B(c_R, R), c_R \in \mathbb{R}^n\}. \end{aligned}$$

Given $0 \leq \alpha \leq 1$, an admissible covering is an α -covering of \mathbb{R}^n if $|Q| \sim (1+|x|^2)^{\alpha n/2}$ (uniformly) for all $Q \in \mathcal{Q}$ and all $x \in Q$ and where $\sup_{Q \in \mathcal{Q}} R_Q/r_Q \leq K$ for some $K < \infty$. Let \mathcal{Q} be an α -covering of \mathbb{R}^n . A bounded admissible partition of unity of order p (abbreviated p -BAPU) corresponding to \mathcal{Q} is a family of smooth functions $\{\psi_Q\}_{Q \in \mathcal{Q}}$ satisfying

$$\begin{aligned} \psi_Q : \mathbb{R}^n &\rightarrow [0, 1], \quad \text{supp } \psi_Q \subset Q, \\ \sum_{Q \in \mathcal{Q}} \psi_Q(\xi) &\equiv 1, \quad \xi \in \mathbb{R}^n, \\ \sup_{Q \in \mathcal{Q}} |Q|^{1/p-1} \|\mathcal{F}^{-1} \psi_Q\|_{L^p} &< \infty \end{aligned}$$

where \mathcal{F} denotes the Fourier transform.

For any $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq \alpha \leq 1$ the α -modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ is the space of all tempered distributions f for which the following norm

$$\|f\|_{M_{p,q}^{s,\alpha}} = \begin{cases} \left(\sum_{Q \in \mathcal{Q}} (1 + |\xi_Q|^2)^{qs/2} \|\mathcal{F}^{-1} \psi_Q \mathcal{F} f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} & \text{if } 1 \leq q < \infty \\ \sup_{Q \in \mathcal{Q}} (1 + |\xi_Q|^2)^{s/2} \|\mathcal{F}^{-1} \psi_Q \mathcal{F} f\|_{L^p} & \text{if } q = \infty \end{cases}$$

is finite, where $\{\xi_Q \in Q : Q \in \mathcal{Q}\}$ is an arbitrary sequence. One shows that this definition is independent of an α -covering \mathcal{Q} and of p -BAPU.

The following embedding results for α -modulation spaces are known. Suppose that $\alpha_1 < \alpha_2 < 1$ and $1 \leq p \leq \infty$. Then

$$M_{p,1}^{s,\alpha_1} \subset M_{p,1}^{s,\alpha_2}$$

and, in particular, for any $\alpha < 1$ and $1 \leq p \leq \infty$ we have

$$M_{p,1}^{s,\alpha} \subset B_{p,1}^s$$

The proofs of these results can be found e.g. in [11]; see Thm. 4.1 and Thm. 4.2.

We next proceed to choose the initial vorticity ω_0 in (1.1). Given any radial bump function $0 \leq \varphi \leq 1$ with support in $B(0, 1/4)$ define

$$(2.1) \quad \varphi_0(x_1, x_2) = \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \varphi(x_1 - \varepsilon_1, x_2 - \varepsilon_2).$$

Fix a positive integer $N_0 \in \mathbb{Z}_+$ and for any $M \gg 1$ let

$$(2.2) \quad \omega_0(x) = M^{-2} N^{-\frac{1}{p}} \sum_{N_0 \leq k \leq N_0 + N} \varphi_k(x),$$

where $1 < p < \infty$, $N = 1, 2, 3, \dots$ and where

$$\varphi_k(x) = 2^{(-1+\frac{2}{p})k} \varphi_0(2^k x).$$

Clearly, the function φ_0 is odd in both variables x_1, x_2 and for any $k \geq 1$ the supports of φ_k are disjoint and contained in a bounded set

$$(2.3) \quad \text{supp } \varphi_k \subset \bigcup_{\varepsilon_1, \varepsilon_2 = \pm 1} B((\varepsilon_1 2^{-k}, \varepsilon_2 2^{-k}), 2^{-(k+2)}).$$

It is easy to check that $\omega_0 \in W^{1,r}(\mathbb{R}^2) \cap C_c^\infty(\mathbb{R}^2)$. In fact, we have

Lemma 4. *For any $2 < r < \infty$ and any positive integer N , we have*

$$(2.4) \quad \|\omega_0\|_{W^{1,r}} \lesssim M^{-2}$$

Proof. A straightforward calculation is omitted. \square

Let $u = \nabla^\perp \Delta^{-1} \omega \in W^{2,r} \cap C^\infty$ be the associated velocity field and consider its Lagrangian flow $\eta(t)$, i.e., the solution of the initial value problem

$$(2.5) \quad \begin{aligned} \frac{d\eta}{dt}(t, x) &= u(t, \eta(t, x)) \\ \eta(0, x) &= x. \end{aligned}$$

It can be checked that $\eta(t)$ is smooth and preserves the coordinate axes x_1, x_2 as well as the symmetries of the initial vorticity ω_0 in (2.2). In fact, the flow is hyperbolic near the origin (a stagnation point) and we have the following

Proposition 5. *Given $M \gg 1$ we have*

$$\sup_{0 \leq t \leq M^{-3}} \|D\eta(t)\|_\infty > M$$

for any sufficiently large integer $N > 0$ in (2.2) and any $2 < r < \infty$ sufficiently close to 2.

Proof. See [18]; Prop. 6. \square

In order to proceed we need the following simple comparison result for the derivatives of solutions of the Lagrangian flow equations.

Lemma 6. *Let u and v be smooth divergence-free vector fields on \mathbb{R}^2 . If η and $\tilde{\eta}$ are the solutions of (2.5) corresponding to u and $u + v$ respectively, then*

$$\sup_{0 \leq t \leq 1} \sum_{i=0,1} \|D^i \eta(t) - D^i \tilde{\eta}(t)\|_\infty \leq C \sup_{0 \leq t \leq 1} \sum_{i=0,1} \|D^i v(t)\|_\infty$$

for some $C > 0$ depending only on the L^∞ norms of u and its derivatives.

Proof. Follows at once by applying Gronwall's inequality to the equation satisfied by the difference $\eta(t) - \tilde{\eta}(t)$. \square

3. THE PROOF OF THE MAIN THEOREM

As in the previous section let $\omega(t) \in W^{1,r} \cap C^\infty$ be the solution of the vorticity equations (1.1) with the initial condition (2.2) and let $\eta(t)$ be the Lagrangian flow of the velocity field $u = \nabla^\perp \Delta^{-1} \omega$ as above. Our main goal in this section will be to prove

Theorem 7. *Let $r > 2$. Assume that the incompressible Euler equations are well-posed in the α -modulation space $M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2)$ for any $p \geq 2$, $q \geq 1$, $0 < \alpha \leq 1$ and $0 < \sigma < 1$ in the sense of Hadamard. Moreover, assume that $M_{p,q}^{1+\sigma,\alpha}$ is topologically embedded in $C^1(\mathbb{R}^2)$. Then there exist a $T > 0$ and a sequence $\omega_{0,n}$ in C_c^∞ with the following properties.*

1. *There is a constant $C > 0$ such that $\|\omega_{0,n}\|_{W^{1,r}} \leq C$ for sufficiently large positive integers n .*
2. *For any $M \gg 1$ there is $0 < t_0 \leq T$ such that $\|\omega_n(t_0)\|_{W^{1,r}} \geq M^{1/3}$ for sufficiently large n and for all r sufficiently close to 2.*

Since Hadamard's notion entails continuity of the data-to-solution map we deduce from Theorem 7 that continuity cannot hold in $M_{p,q}^{1+s,\alpha}(\mathbb{R}^2)$ or else we get a contradiction with the following result.

Theorem (Kato-Ponce [14]). *Let $1 < r < \infty$ and $s > 1 + 2/r$. For any $\omega_0 \in W^{s-1,r}(\mathbb{R}^2)$ and any $T > 0$ there exists a constant $K = K(T, \|\omega_0\|_{W^{s-1,r}}) > 0$ such that*

$$\sup_{0 \leq t \leq T} \|\omega(t)\|_{W^{s-1,r}} \leq K.$$

Our Main Theorem will be a direct consequence of Theorem 7.

Proof of Theorem 7. Given any large number $M \gg 1$ pick $T \leq M^{-3}$. Observe that if $\|\omega_0(t_0)\|_{W^{1,r}} > M^{1/3}$ for some $0 < t_0 \leq M^{-3}$ then there is nothing to prove and therefore we may assume that

$$(3.1) \quad \|\omega(t)\|_{W^{1,r}} \leq M^{1/3}, \quad 0 \leq t \leq M^{-3}.$$

Next, by Proposition 5 we can find $0 \leq t_0 \leq M^{-3}$ and a point $x^* = (x_1^*, x_2^*)$ in \mathbb{R}^2 for which the absolute value of one of the entries in the Jacobi matrix $D\eta(t_0, x^*)$ is at least as large as M . Because the velocity field u is in $W^{2,r}$ so is the associated Lagrangian flow² and hence by continuity in some sufficiently small δ -neighbourhood of x^* we have e.g.,

$$(3.2) \quad \left| \frac{\partial \eta_2}{\partial x_2}(t_0, x) \right| > M \quad \text{for all } |x - x^*| < \delta.$$

We proceed to construct a sequence of high-frequency perturbations of the initial vorticity in $W^{1,r}$. To this end we choose a smooth bump function $0 \leq \hat{\chi} \leq 1$ with compact support in the unit ball $B(0, 1)$ in the Fourier space and normalized by $\int_{\mathbb{R}^2} \hat{\chi}(\xi) d\xi = 1$. Using this function we set

$$(3.3) \quad \hat{\rho}(\xi) = \hat{\chi}(\xi - \xi_0) + \hat{\chi}(\xi + \xi_0), \quad \xi \in \mathbb{R}^2, \quad \xi_0 = (2, 0)$$

so that $\text{supp } \hat{\rho} \subset B(-\xi_0, 1) \cup B(\xi_0, 1)$ with

$$(3.4) \quad \rho(0) = \int_{\mathbb{R}^2} \hat{\rho}(\xi) d\xi = 2$$

and observe that for any $a > 4$ we have

$$(3.5) \quad \text{supp } \hat{\rho}(\cdot \pm a, \cdot) \cap B(0, 1) = \emptyset.$$

Define

$$(3.6) \quad \beta_{k,\lambda}^{\tilde{\alpha},r}(x) = \frac{\lambda^{-1+\frac{2}{r}}}{k^{1-\tilde{\alpha}}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \rho(\lambda(x - x_\epsilon^*)) \sin kx_1, \quad k \in \mathbb{Z}^+, \lambda > 0$$

where $x_\epsilon^* = (\varepsilon_1 x_1^*, \varepsilon_2 x_2^*)$ and $\lambda > 0$ and $0 < \tilde{\alpha} < 1$ will be further specified below.

²E.g., by the wellposedness theory of [14].

Remark 8. Note that the parameter λ in (3.6) relates to the speed with which the support of the function ρ is spreading in the Fourier space while k expresses the speed of its translation. In the standard modulation space $M_{\infty,1}^{1+\sigma,0}(\mathbb{R}^2)$ one would need to set $\lambda = 1$ but in that case the spreading speed of the support of ρ (its shrinking speed in physical space) is zero and hence the arguments we apply in the present paper break down. Therefore, the case of the standard modulation space $M_{\infty,1}^{1+\sigma,0}(\mathbb{R}^2)$ remains an open problem.

Before defining a suitable perturbation of ω_0 we need to derive several estimates for $\beta_{k,\lambda}^{\tilde{\alpha},r}$ which we collect in the following lemma.

Lemma 9. *Let $2 \leq p \leq \infty$, $2 < r < \infty$ and $0 < \sigma < 1$. For any $k \in \mathbb{Z}^+$ and $\lambda > 0$ sufficiently large, we have*

1. $\|\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{W^{1,r}} \lesssim k^{-1+\tilde{\alpha}} + k^{\tilde{\alpha}}\lambda^{-1}$
2. $\|\Delta^{1/2+\sigma}\partial_j\Delta^{-1}\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^p} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1+2(1/r-1/p)}(\lambda^\sigma + k^\sigma)$
3. $\|\partial_j\Delta^{-1}\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^p} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1+2(1/r-1/p)}$

where $j = 1, 2$.

Proof. We need to compute the L^r norms of $\beta_{k,\lambda}^{\tilde{\alpha},r}$ and its first derivative $\partial_i\beta_{k,\lambda}^{\tilde{\alpha},r}$. By the triangle inequality and the fact that $\hat{\rho}$ has compact support we have

$$\|\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^r} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1} \sum_{\varepsilon_1, \varepsilon_2} \left(\int_{\mathbb{R}^2} |\rho(\lambda(x - x_\varepsilon^*))|^r \lambda^2 dx \right)^{1/r} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1}.$$

For the first derivatives, we have

$$\begin{aligned} \left\| \frac{\partial \beta_{k,\lambda}^{\tilde{\alpha},r}}{\partial x_1} \right\|_{L^r} &\lesssim k^{-1+\tilde{\alpha}}\lambda^{2/r} \sum_{\varepsilon_1, \varepsilon_2} \left\| \frac{\partial \rho}{\partial x_1}(\lambda(\cdot - x_\varepsilon^*)) \right\|_{L^r} + k^{\tilde{\alpha}}\lambda^{-1+2/r} \sum_{\varepsilon_1, \varepsilon_2} \|\rho(\lambda(\cdot - x_\varepsilon^*))\|_{L^r} \\ &\simeq k^{-1+\tilde{\alpha}} \left\| \frac{\partial \rho}{\partial x_1} \right\|_{L^r} + k^{\tilde{\alpha}}\lambda^{-1} \|\rho\|_{L^r} \lesssim k^{-1+\tilde{\alpha}} + k^{\tilde{\alpha}}\lambda^{-1} \end{aligned}$$

and similarly

$$\left\| \frac{\partial \beta_{k,\lambda}^{\tilde{\alpha},r}}{\partial x_2} \right\|_{L^r} \lesssim k^{-1+\tilde{\alpha}} \left\| \frac{\partial \rho}{\partial x_2} \right\|_{L^r} \lesssim k^{-1+\tilde{\alpha}}.$$

Combining these estimates gives the bound for $\|\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{W^{1,r}}$.

In order to derive the estimates in the remaining cases it will be convenient to use the Fourier transform

$$(3.7) \quad \hat{\beta}_{k,\lambda}^{\tilde{\alpha},r}(\xi) = \frac{\lambda^{-1+2/r}}{k^{1-\tilde{\alpha}}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \sum_{j=1,2} \varepsilon_1 \varepsilon_2 \frac{(-1)^{j+1}}{2i\lambda^2} \hat{\rho}(\lambda^{-1}\xi_j^k) e^{-2\pi i \langle x_\varepsilon^*, \xi_j^k \rangle}$$

where $\xi_j^k = (\xi_1 + \frac{(-1)^j}{2\pi}k, \xi_2)$. Let p' be the conjugate exponent to p . Applying the Hausdorff-Young inequality we obtain

$$\begin{aligned} \|\Delta^{\frac{1+\sigma}{2}}\partial_j\Delta^{-1}\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^p} &\lesssim \| |\cdot|^\sigma \hat{\beta}_{k,\lambda}^{\tilde{\alpha},r} \|_{L^{p'}} \\ &\lesssim k^{-1+\tilde{\alpha}}\lambda^{-1+2/r} \sum_{j=1,2} \left(\int_{\mathbb{R}^2} \lambda^{-2p'} |\xi|^{\sigma p'} |\hat{\rho}(\lambda^{-1}\xi_j^k)|^{p'} d\xi \right)^{1/p'} \end{aligned}$$

and changing the variables we further estimate by

$$\begin{aligned}
&\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{2}{r}-2(1-\frac{1}{p'})} \sum_{j=1,2} \left(\int_{\mathbb{R}^2} \left(\left(\eta_1 - \frac{(-1)^j}{2\pi} k \right)^2 + \eta_2^2 \right)^{\frac{\sigma p'}{2}} |\hat{\rho}(\lambda^{-1}\eta)|^{p'} \frac{d\eta}{\lambda^2} \right)^{1/p'} \\
&\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{2}{r}-\frac{2}{p}} \sum_{j=1,2} \left(\int_{\mathbb{R}^2} \left(\left(\lambda\zeta_1 - \frac{(-1)^j}{2\pi} k \right)^2 + (\lambda\zeta_2)^2 \right)^{\frac{\sigma p'}{2}} |\hat{\rho}(\zeta)|^{p'} d\zeta \right)^{1/p'} \\
&\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+2(\frac{1}{r}-\frac{1}{p})} (\lambda^\sigma + k^\sigma)
\end{aligned}$$

for any $\sigma \geq 0$.

By the same calculation as above, we also have

$$\begin{aligned}
\|\partial_j \Delta^{-1} \beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^p} &\lesssim \|\cdot\|^{-1} \hat{\beta}_{k,\lambda}\|_{L^{p'}} \\
&\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{2}{r}-\frac{2}{p}} \sum_{j=1,2} \left(\int_{\mathbb{R}^2} \left(\left(\lambda\zeta_1 - \frac{(-1)^j}{2\pi} k \right)^2 + (\lambda\zeta_2)^2 \right)^{-\frac{p'}{2}} |\hat{\rho}(\zeta)|^{p'} d\zeta \right)^{1/p'} \\
&\lesssim k^{-1+\tilde{\alpha}} \lambda^{-2+2(\frac{1}{r}-\frac{1}{p})}
\end{aligned}$$

for sufficiently large k and λ . \square

From now on we will restrict to the case

$$(3.8) \quad \lambda = k^{\tilde{\alpha}}, \quad k = n \quad \text{and} \quad 0 < \tilde{\alpha} \leq \alpha \leq 1$$

and observe that it is possible to choose the integers n are sufficiently large so that, in particular, the assumptions of the previous lemma hold. Let $\beta_n = \beta_{k,\lambda}^{\tilde{\alpha},r}$ and define a sequence of initial vorticities by

$$(3.9) \quad \omega_{0,n}(x) = \omega_0(x) + \beta_n(x), \quad n \gg 10.$$

Combining the first part of Lemma 9 with equation (2.4) of Lemma 4 we find that $\omega_{0,n}$ belong to $W^{1,r}$, namely

$$(3.10) \quad \|\omega_{0,n}\|_{W^{1,r}} \leq \|\omega_0\|_{W^{1,r}} + \|\beta_n\|_{W^{1,r}} \lesssim 1$$

for any sufficiently large n . This proves the first assertion of Theorem 7.

Denote by $\omega_n(t)$ the sequence of vorticity solutions of (1.1) with initial data $\omega_{0,n}$ and as before let $\eta_n(t)$ be the Lagrangian flows of the corresponding velocity fields $u_n = \nabla^\perp \Delta^{-1} \omega_n$. The following lemma will be crucial in what follows.

Lemma 10. *Let $0 < \sigma < 1$, $0 < \alpha \leq 1$ and $2 \leq p \leq \infty$. For any $1 \leq q \leq \infty$ we have*

$$\|\nabla^\perp \Delta^{-1} \beta_n\|_{M_{p,q}^{1+\sigma,\alpha}} \simeq \|\nabla^\perp \Delta^{-1} \beta_n\|_{L^p} + \|\Delta^{\frac{1+\sigma}{2}} \nabla^\perp \Delta^{-1} \beta_n\|_{L^p}$$

for any sufficiently large $n \in \mathbb{Z}^+$.

Proof. From (3.5) and (3.7) we see that for any sufficiently large integer $n \in \mathbb{Z}^+$ the subsets $\text{supp } \hat{\beta}_n$ and $B(0,1)$ are disjoint. Thus, it suffices to consider the case $\sigma = 0$, that is

$$\|\beta_n\|_{L^p} \simeq \|\beta_n\|_{M_{p,q}^{0,\alpha}}.$$

Let \mathcal{Q} be an admissible α -covering by sets of size $|Q| \sim (1 + |x|^2)^\alpha$. Using (3.7), (3.8) and the fact that $\text{supp } \hat{\rho} \subset B(0, 3)$ we find

$$\text{supp } \hat{\beta}_{n=2^j} \subset B((2^j, 0), 2^{\tilde{\alpha}j}) \subset B((2^j, 0), 2^{\alpha j})$$

so that for any $j \in \mathbb{Z}_+$ there is a $Q \in \mathcal{Q}$ with $\text{supp } \hat{\beta}_{2^j} \subset Q$ and we have

$$\|\beta_{2^j}\|_{M_{p,q}^{0,\alpha}} = \left(\sum_{Q \in \mathcal{Q}} \|\mathcal{F}^{-1} \psi_Q \mathcal{F} \beta_{2^j}\|_p^q \right)^{1/q} \simeq \|\beta_{2^j}\|_{L^p}$$

for $q < \infty$. Note that the case $q = \infty$ is analogous. \square

Suppose now that the data-to-solution map for the Euler equations (E) is continuous from bounded subsets in $M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2)$ into $C([0, 1], M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2))$. Choose $0 < \tilde{\alpha} \leq \alpha$ so that

$$-1 + \sigma + 2\tilde{\alpha}(1/r - 1/p) < 0.$$

Then, from estimates 2 and 3 of Lemma 9 we have

$$\|\nabla^\perp \Delta^{-1} \beta_n\|_{L^p} + \|\Delta^{\frac{1+\sigma}{2}} \nabla^\perp \Delta^{-1} \beta_n\|_{L^p} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

where β_n is the perturbation sequence defined in (3.6) and combining (3.9) with Lemma 10 we obtain

$$(3.11) \quad \|\nabla^\perp \Delta^{-1}(\omega_{0,n} - \omega_0)\|_{M_{p,q}^{1+\sigma,\alpha}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The continuity assumption on the solution map and (3.11) now imply

$$(3.12) \quad \sup_{0 \leq t \leq T} \|\nabla^\perp \Delta^{-1}(\omega_n(t) - \omega(t))\|_{M_{p,q}^{1+\sigma,\alpha}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

from which, using the embedding assumption $M_{p,q}^{1+\sigma} \subset C^1$ and Lemma 6, we obtain

$$(3.13) \quad \sup_{0 \leq t \leq T} \sum_{i=0,1} \|D^i \eta_n(t) - D^i \eta(t)\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

where as before $\eta(t)$ is the Lagrangian flow of the (smooth) divergence-free vector field $u = \nabla^\perp \Delta^{-1} \omega$ of Proposition 5 and $\eta_n(t)$ is the flow of $u_n = \nabla^\perp \Delta^{-1} \omega_n$ whose initial vorticities are given by (2.2) and (3.9) respectively. The rest of the argument is completely analogous to the proof of Theorem 3 in [18]. Thus we omit the details. \square

4. A DIRECT PROOF FOR $C^{1+\sigma}(\mathbb{R}^3)$ BASED ON SHEAR FLOW

In this section we present a short and direct argument showing the loss of continuity of the data-to-solution map of (E) in the classical Hölder space $C^{1+\sigma}$ with $0 < \sigma < 1$. It is inspired by conversations with A. Shnirelman and C. Bardos from whom we learned about the DiPerna-Majda shear flow techniques.

Consider two 3D shear flows

$$u(t, x) = (f(x_2), 0, h(x_1 - tf(x_2))) \quad \text{and} \quad v(t, x) = (g(x_2), 0, h(x_1 - tg(x_2))).$$

Let f, g and h be bounded functions in $C^{1+\sigma}(\mathbb{R}^3)$ with h chosen so that in addition its derivative satisfies

$$h'(x_1) = |x_1|^\sigma \quad \text{for} \quad -2a \leq x_1 \leq 2a$$

where $a = \max\{\sup_{x_1} |f(x_1)|, \sup_{x_1} |g(x_1)|\}$. It is easy to verify that both $u(t)$ and $v(t)$ solve the 3D Euler equations with initial conditions

$$u_0(x) = (f(x_2), 0, h(x_1)) \quad \text{and} \quad v_0(x) = (g(x_2), 0, h(x_1)).$$

Now, given any $\epsilon > 0$ adjust f and g so that

$$\|u_0 - v_0\|_{C^{1+\sigma}} = \|f - g\|_{C^{1+\sigma}} < \epsilon$$

and consider at any time $0 \leq t \leq 1$ the norm of the difference of the corresponding solutions

$$\begin{aligned} \|u(t) - v(t)\|_{C^{1+\sigma}} &= \|f - g\|_{C^{1+\sigma}} + \|h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot))\|_{C^{1+\sigma}} \\ &\geq \|\nabla(h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot)))\|_{C^\sigma} \\ &= \|h'(\cdot - tf(\cdot)) - h'(\cdot - tg(\cdot))\|_{C^\sigma} \end{aligned}$$

which can be further bounded from below by

$$\geq \sup_{\substack{x \neq y \\ x, y \in [-b, b]^2}} \frac{|(|x_1 - tf(x_2)|^\sigma - |x_1 - tg(x_2)|^\sigma) - (|y_1 - tf(y_2)|^\sigma - |y_1 - tg(y_2)|^\sigma)|}{|x - y|^\sigma}$$

where $b = \min\{\sup_{x_1} |f(x_1)|, \sup_{x_1} |g(x_1)|\}$.

Finally, pick $x_2 = y_2 = c$ for some arbitrary constant c so that the expression above becomes

$$\geq \sup_{\substack{x_1 \neq y_1 \\ x, y \in [-b, b]^2}} \frac{|(|x_1 - tf(c)|^\sigma - |x_1 - tg(c)|^\sigma) - (|y_1 - tf(c)|^\sigma - |y_1 - tg(c)|^\sigma)|}{|x_1 - y_1|^\sigma}$$

and evaluate it once again from below by choosing $x_1 = tg(c)$ and $y_1 = tf(c)$ to get the bound

$$\geq \frac{t^\sigma |g(c) - f(c)|^\sigma + t^\sigma |f(c) - g(c)|^\sigma}{t^\sigma |f(c) - g(c)|^\sigma} = 2.$$

This shows local ill-posedness of the 3D Euler equations in $C^{1+\sigma}$ in the Hadamard sense considered here. \square

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REFERENCES

1. H. Bahouri, J. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, New York 2011.
2. C. Bardos and I. Titi, *Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations*, Discrete Cont. Dyn. Syst. ser. **S3** (2010), 185-197.
3. J. Bourgain and D. Li, *Strong ill-posedness of the incompressible Euler equations in borderline Sobolev spaces*, to appear in Invent. math.
4. J. Bourgain and D. Li, *Strong illposedness of the incompressible Euler equation in integer C^m spaces*, preprint arXiv:1405.2847 [math.AP].
5. P. Constantin, *On the Euler equations of incompressible fluids*, Bull. Amer. Math. Soc. **44** (2007), 603-621.
6. A. Cheskidov and R. Shvydkoy, *Ill-posedness of basic equations of fluid dynamics in Besov spaces*, Proc. A.M.S. **138** (2010), 1059-1067.

7. D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math. **92** (1970), 102-163.
8. T. Elgindi and N. Masmoudi, *L^∞ ill-posedness for a class of equations arising in hydrodynamics*, preprint arXiv:1405.2478 [math.AP].
9. H. G. Feichtinger, C. Huang and B. Wang, *Trace operators for modulation, α -modulation and Besov spaces*, Appl. Comput. Harmon. Anal. **30** (2011), 110-127.
10. N. Gyunter, *On the motion of a fluid contained in a given moving vessel*, (Russian), Izvestia AN USSR, Sect. Phys. Math. (1926-8).
11. J. Han and B. Wang, *α -modulation spaces (I) embedding, interpolation and algebra properties*, J. Math. Soc. Japan **66** (2014), 1315-1373.
12. T. Kato and C. Lai, *Nonlinear evolution equations and the Euler flow*, J. Funct. Anal. **56** (1984), 15-28.
13. T. Kato and G. Ponce, *Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbb{R}^2)$* , Rev. Mat. Iberoamericana **2** (1986), 73-88.
14. T. Kato and G. Ponce, *On nonstationary flows of viscous and ideal fluids in $L^p_s(\mathbb{R}^2)$* , Duke Math. J. **55** (1987), 487-499.
15. L. Lichtenstein, *Über einige Existenzprobleme der Hydrodynamik unzusamendruckbarer, reibungsloser Flüssigkeiten und die Helmholtzschen Wirbelsatze*, Math. Zeit. **23** (1925), **26** (1927), **28** (1928), **32** (1930).
16. A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge 2002.
17. G. Misiolek and T. Yoneda, *Ill-posedness examples for the quasi-geostrophic and the Euler equations*, Analysis, geometry and quantum field theory, Contemp. Math. **584**, Amer. Math. Soc., Providence, RI, 2012, 251-258.
18. G. Misiolek and T. Yoneda, *Local ill-posedness of the incompressible Euler equations in C^1 and $B^1_{\infty,1}$* , preprint arXiv:1405.1943 [math.AP] and arXiv:1405.4933 [math.AP] (2014).
19. A. Morgulis, A. Shnirelman and V. Yudovich, *Loss of smoothness and inherent instability of 2D inviscid fluid flows*, Comm. Partial Differential Eqns **33** (2008), 943-968.
20. H. Pak and Y. Park, *Existence of solution for the Euler equations in a critical Besov space $B^1_{\infty,1}(\mathbb{R}^n)$* , Comm. Partial Differential Equations **29** (2004), 1149-1166.
21. R. Takada, *Counterexamples of commutator estimates in the Besov and the Triebel-Lizorkin spaces related to the Euler equations*, SIAM J. Math. Anal. **42** (2010), 2473-2483.
22. M. Vishik, *Hydrodynamics in Besov spaces*, Arch. Rational Mech. Anal. **145** (1998), 197-214.
23. W. Wolibner, *Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long*, Math. Z. **37** (1933), 698-726.
24. V. Yudovich, *Non-stationary flow of an incompressible liquid*, Zh. Vychis. Mat. Mat. Fiz. **3** (1963), 1032-1066.

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